

# Relative Invariants, Ideal Classes and Quasi-Canonical Modules of Modular Rings of Invariants

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## ABSTRACT

We describe “quasi canonical modules” for modular invariant rings  $R$  of finite group actions on factorial Gorenstein domains. From this we derive a general “quasi Gorenstein criterion” in terms of certain 1-cocycles. This generalizes a recent result of A. Braun for linear group actions on polynomial rings, which itself generalizes a classical result of Watanabe for non-modular invariant rings.

We use an explicit classification of all reflexive rank one  $R$ -modules, which is given in terms of the class group of  $R$ , or in terms of  $R$ -semi-invariants. This result is implicitly contained in a paper of Nakajima ([6]).

## 1. Introduction

Let  $k$  be a field,  $V$  a finite dimensional  $k$ -vector space of dimension  $n$ ,  $G \subseteq \mathrm{GL}(V)$  a finite group and  $A := \mathrm{Sym}(V^*) \cong k[x_1, \dots, x_n]$ , the symmetric algebra over the dual space  $V^*$  with its canonical  $G$ -action and ring of invariants  $R := A^G := \{a \in A \mid ga = a \ \forall g \in G\}$ .

A classical result of K. Watanabe states that if  $p = \mathrm{char}(k)$  does not divide  $|G|$ , then  $A^G$  is Gorenstein if  $G \subseteq \mathrm{SL}(V)$ . If moreover  $G$  contains no pseudo-reflection, then the converse holds, i.e. if  $A^G$  is Gorenstein, then  $G \subseteq \mathrm{SL}(V)$  ([7], [8]). In the recent paper [2], A. Braun proved an analogue of this result for the modular case, where the characteristic of  $k$  is allowed to divide the group order. Consider the following

**Hypothesis** ( $\mathcal{NR}$ ) : The group  $G \subseteq \mathrm{GL}(V)$  contains no pseudo-reflection (neither diagonalizable nor transvection).

Then Braun proved the following result:

**THEOREM 1.1.** [2] *Let  $k$  be an arbitrary field and suppose the Hypothesis ( $\mathcal{NR}$ ) holds. Then the following are equivalent:*

- (i)  $G \subseteq \mathrm{SL}(V)$ ;
- (ii)  $A^G \cong \mathrm{Hom}_C(A^G, C)$  for every polynomial ring  $C \subseteq A^G$  such that  $A^G$  is a finitely generated  $C$ -module and the homogeneous generators of  $C$  have degrees divisible by  $|G|$ .

From this he deduces that if  $G$  satisfies Hypothesis ( $\mathcal{NR}$ ), then the Cohen-Macaulay and Gorenstein loci of  $A^G$  coincide and if  $A^G$  is Cohen-Macaulay it is also Gorenstein. He also obtains a modular version of the converse: If  $G$  satisfies Hypothesis ( $\mathcal{NR}$ ) and  $A^G$  is Gorenstein, then  $G$  is contained in  $\mathrm{SL}(V)$ .

In this paper we generalize Braun’s results in two ways: firstly we avoid Hypothesis ( $\mathcal{NR}$ )

altogether. Secondly we neither assume  $A$  to be a polynomial ring nor that the parameter algebra  $C$  is chosen in any particular way. Instead, our main result applies, whenever  $A$  is a (not necessarily graded)  $k$ -algebra, which is also a factorial domain with unit group  $U(A) = U(k)$ . It is remarkable that Braun's proof employs important techniques from the theory of non-commutative Frobenius and symmetric algebras. The current paper grew out of our attempt to understand these methods in detail, in particular with an eye on possible future applications in non-commutative invariant theory. Nevertheless, it turned out, that Braun's result as well as our generalization, can be obtained wholly within the "world of commutative algebra", by combining Braun's ideas with information hidden in the proofs of a classical paper by Nakajima [6].

To formulate our main result we need the following definitions and notation:

Let  $A$  be a  $k$ -algebra which is also a factorial domain with unit group  $U(A) = U(k)$  and let  $G \subseteq \text{Aut}(A)$  be a finite group. We do not assume that  $G$  acts trivially on  $k$ , so  $k' := k^G$  can be a proper subfield of  $k$ .

DEFINITION 1. Let  $\lambda \in Z^1(G, U(A))$  be a 1-cocycle, i.e.  $\lambda : G \rightarrow U(A)$  with

$$\lambda(gh) = \lambda(g) \cdot g(\lambda(h)) \quad \forall g, h \in G.$$

Then we define  $A_\lambda := \{a \in A \mid g(a) := \lambda(g)a\}$ , the  $R$ -module of relative  $\lambda$ -invariants, or  $\lambda$ -semi invariants in  $A$ .

DEFINITION 2. Let  $\mathcal{P}$  be a commutative Gorenstein ring and  $B$  a commutative  $\mathcal{P}$ -algebra such that  ${}_{\mathcal{P}}B$  is finite. Then we call the  $B$ -module  $\text{Hom}_{\mathcal{P}}(B, \mathcal{P})$  the *quasi-canonical module* of  $B$  and we call  $B$  *quasi-Gorenstein* (w.r.t.  $\mathcal{P}$ ), if  $\text{Hom}_{\mathcal{P}}(B, \mathcal{P}) \cong B$  as  $B$ -modules (in other words, if  $B$  is a symmetric  $\mathcal{P}$ -algebra).

REMARK 1. If  $B$  is a graded connected  $k$ -algebra and  $\mathcal{P}$  a polynomial ring, generated by a homogeneous system of parameters, then  $B$  is a Cohen-Macaulay ring, if and only if  ${}_{\mathcal{P}}B$  is free. If  $B$  is Cohen-Macaulay, then it is well known that  $\omega_B := \text{Hom}_{\mathcal{P}}(B, \mathcal{P})$  is a canonical module of  $B$  and  $B$  is Gorenstein, if and only if  $B \cong \omega_B$ .

Let  $W := W(G) \trianglelefteq G$  be the normal subgroup generated by generalized reflections (see Definition 5) and let  $\mathcal{F}$  be any parameter  $k'$ -subalgebra  $\mathcal{F} \subseteq R := A^G \subseteq S := A^W \subseteq A$ . Although not explicitly stated in [6], the following facts are implicit in the proofs of that paper:

- (1) The class group  $\mathcal{C}_R$  of  $R$  is isomorphic to the subgroup  $\tilde{H}$  of  $H^1(G, U(A))$ , defined by  $\tilde{H} := \{\rho \in H^1(G, U(A)) \mid \text{res}_{I_Q}(\rho) = 1 \text{ in } H^1(I_Q, U(A_Q)), \forall Q \in \text{Spec}_1(A)\}$ . (see Theorem 3.4).
- (2) There are explicit bijections between the following sets:
  - the divisor class group  $\mathcal{C}_R$ ;
  - the set of iso types of finitely generated reflexive  $R$ -modules of rank one;
  - the set of iso types of  $R$  modules of semi-invariants  $A_\chi$  with  $\chi \in Z^1(G, U(A))$ .
- (3) If  $\chi \in Z^1(G/W, U(A))$ , then  $A_\chi \cong R \iff [\chi] = 1 \in H^1(G/W, U(A))$ .

We can now state the main result of this paper:

THEOREM 1.2. The rings  $S$  and  $A$  are quasi-Gorenstein  $\mathcal{F}$ -algebras with

$$\text{Hom}_{\mathcal{F}}(S, \mathcal{F}) = S \cdot \theta_S \cong S, \quad \text{Hom}_{\mathcal{F}}(A, \mathcal{F}) = A \cdot \theta_A \cong A \text{ and } \mathcal{D}_{A,S}^{-1} \cong \text{Hom}_S(A, S) = A \cdot \theta_{A,S}.$$

Here  $\mathcal{D}_{A,R} = \mathcal{D}_{A,S}$  is the Dedekind-different, which is a principal ideal in  $A$  (see Definition 6). Let  $\chi_S \in Z^1(G/W, U(k))$  and  $\chi_A, \chi_{A,S} \in Z^1(G, U(k))$  be the “eigen-characters” of  $\theta_S$ ,  $\theta_A$  and  $\theta_{A,S}$ , respectively. Then  $\chi_S = \chi_A \cdot \chi_{A,S}^{-1}$  and  $\text{Hom}_{\mathcal{F}}(R, \mathcal{F})$  is isomorphic to the  $R$ -module of semi-invariants  $S_{\chi_S^{-1}} = A_{\chi_S^{-1}}$ . In particular the following hold:

- (i) The quasi-canonical  $R$ -module  $\text{Hom}_{\mathcal{F}}(R, \mathcal{F})$  is isomorphic to a divisorial ideal  $I \trianglelefteq R$ , with  $\text{ch}(\text{cl}(I)) = [\chi_S] = [\chi_A/\chi_{A,S}]$ , where  $\text{ch} : \mathcal{C}_R \rightarrow H^1(G/W, U(k))$  is the isomorphism of Corollary 3.12.
- (ii) The following are equivalent:
  - (a) The ring  $R$  is quasi-Gorenstein;
  - (b)  $[\chi_S] = 1 \in H^1(G/W, U(k))$ .
  - (c)  $[\chi_{A,S}] = [\chi_A] \in H^1(G, U(k))$ .

REMARK 2.

In the special case, where  $A$  is a polynomial ring with  $k$ -linear  $G$  action, the equivalence of (ii) (a) and (c) also appears in a paper by A Broer ([3]).

COROLLARY 1.3. If  $[\chi_S] = 1 \in H^1(G/W, U(k))$ , then the Cohen-Macaulay and Gorenstein loci of  $R$  coincide.

If  $\text{char}(k) = p > 0$ , set  $\tilde{W} := \langle W, P^g \mid g \in G \rangle$  with  $P$  a Sylow  $p$ -subgroup of  $G$ . In other words,  $\tilde{W} \trianglelefteq G$  is the normal subgroup generated by all reflections on  $A$  and all elements of order a power of  $p$ . We obtain:

COROLLARY 1.4. If  $G$  acts trivially on  $k$ , then  $H^1(G/W, U(k)) = \text{Hom}(G/W, U(k)) = \text{Hom}(G/\tilde{W}, U(k))$  and Theorem 1.2 also holds with  $W$  and  $S$  replaced by  $\tilde{W}$  and  $\tilde{S} := A^{\tilde{W}}$ . In particular  $\tilde{S}$  is a factorial domain and quasi-Gorenstein and  $R$  is quasi-Gorenstein  $\iff \chi_{\tilde{S}} = 1$ .

Assume for the moment that Hypothesis  $(\mathcal{NR})$  holds, then  $W = 1$  and  $A = S$  with  $[\chi_{A,S}] = 1$ . Hence in this case  $R$  is quasi-Gorenstein, if and only if  $[\chi_A] = 1$ . If moreover  $A = \text{Sym}(V^*)$  with  $G \subseteq \text{GL}_k(V)$ , then  $[\chi_A] = \chi_A = \det^{-1}$  (see Remark 4) and we recover Braun’s result (and Watanabe’s for  $\text{char}(k) \nmid |G|$ ). More generally:

COROLLARY 1.5. Assume that  $A = \text{Sym}(V^*)$  and  $S := A^W$  is Gorenstein (e.g a polynomial ring). Assume moreover that  $\chi_S = 1$  (note that  $\chi_S \in \text{Hom}(G, U(k))$  here). Then  $R = A^G$  is Gorenstein, if it is Cohen-Macaulay.

It is known by a result of Serre ([1]) that if  $\text{Sym}(V^*)^H$  is a polynomial ring for finite  $H \subseteq \text{GL}_k(V)$ , then  $H = W(H)$ . Unfortunately the converse is false, so the hypothesis of the above Corollary is not automatic. If however it is satisfied, then the character  $\chi_S$  can be explicitly described in terms of the  $G/W$  action on the homogeneous generators of  $A^W$  (see Section 5).

## 2. The divisor class group and reflexive modules of rank one

In this section we collect some definitions and results from [6], including some information which is implicitly contained via arguments and proofs, but not explicitly stated there. In such a case we include short proofs. Let  $A$  be a Krull domain with quotient field  $\mathbb{L} := \text{Quot}(A)$ . Let  $\text{Spec}_1(A) := \{0 \neq P \in \text{Spec}(A) \mid \text{ht}(P) = 1\}$  then for every  $P \in \text{Spec}_1(A)$ , the localization  $A_P$  is a discrete valuation ring and by definition

- (1)  $A = \bigcap_{P \in \text{Spec}_1(A)} A_P$ ;
- (2) for every  $0 \neq \ell \in \mathbb{L}$  the set  $\{P \in \text{Spec}_1(A) \mid \nu_P(\ell) \neq 0\}$  is finite.

Let  $\mathcal{D}_A$  denote the divisor group of  $A$ , i.e. the free abelian group with basis  $\text{Spec}_1(A)$ :

$$\mathcal{D}_A := \bigoplus_{P \in \text{Spec}_1(A)} \mathbb{Z} \text{div}(P).$$

Let  $0 \neq J \triangleleft A$  be an ideal with  $0 \neq j \in J$ . Then  $\nu_P(J) \in \mathbb{Z}$  is defined by  $JA_P = P^{\nu_P(J)} A_P$ , hence  $\nu_P(j) := \nu_P(jA_P) \geq \nu_P(J) \geq 0$ , and it follows that  $\nu_Q(J) = 0$  for almost all  $Q \in \text{Spec}_1(A)$ . If  $I \subseteq \mathbb{L}$  is a fractional ideal, then  $\ell I \triangleleft A$  for some  $\ell \in A$ , hence again  $\nu_Q(I) = 0$  for almost all  $Q \in \text{Spec}_1(A)$  and one defines

$$\text{div}(I) := \sum_{P \in \text{Spec}_1(A)} \nu_P(I) \text{div}(P).$$

With  $\mathcal{H}_A$  we denote the group of principal fractional ideals in  $A$ , then the map  $\text{div}$  embeds  $\mathcal{H}_A$  into  $\mathcal{D}_A$  as a subgroup with quotient group  $\mathcal{C}_A := \mathcal{D}_A / \mathcal{H}_A$ , the *divisor class group* of  $A$ .

**DEFINITION 3.** Let  $R \subseteq A$  be a subring. For ideals  $I \triangleleft R$  or  $J \triangleleft A$  we denote with  $\overline{I}$  and  $\overline{J}$  the corresponding divisorial closures, i.e.

$$\overline{I} = \bigcap_{\substack{Rr \triangleleft R \\ I \subseteq Rr}} Rr, \text{ and } \overline{J} = \bigcap_{J \subseteq Aa} Aa.$$

**LEMMA 2.1.** Let  $R \subseteq A$  be a subring with  $\text{Quot}(R) \cap A = R$ , and let  $I \triangleleft R$  and  $J \triangleleft A$  be ideals, then

1.  $\overline{IA} \cap R = \overline{I}$ ;
2.  $\overline{J} = d_J A$  with  $d_J \in A \iff d_J := \text{gcd}(J)$  exists in  $A$ .
3.  $\overline{IA} = d_I A$  with  $d_I \in A \iff d_I := \text{gcd}(I) := \text{gcd}\{r \in I\}$  (taken inside  $A$ ) exists.
4. For  $a \in A$ :  $\overline{aJ} \subseteq \overline{J} \cdot a$  with equality, if  $J$  and  $aJ$  have a gcd in  $A$  (the latter is then  $ad_J$ ).
5. For  $a \in A$  and divisorial ideal  $J \triangleleft A$ ,  $aJ \triangleleft A$  is divisorial.
6. If every subset of  $A$  has a gcd (e.g. if  $A$  is a factorial domain), then for any ideals  $J, K \triangleleft A$ :  $\overline{J \cdot K} = \overline{J} \cdot \overline{K}$ .

**Proof:** By the assumption on  $\text{Quot}(R)$  we have  $rA \cap R = rR$  for every  $r \in R$ .

1.: Let  $\overline{I} = \bigcap_{\substack{rR \subseteq R \\ r \in R}} rR$ ,  $\overline{IA} = \bigcap_{\substack{IA \subseteq Aa \\ a \in A}} Aa$ , and  $x \in \overline{IA} \cap R$ . Then  $I \subseteq rR$  implies  $\overline{IA} \subseteq rA$ , so  $x \in rA \cap R = rR$ . It follows that  $x \in \overline{I}$  and  $\overline{I} \subseteq \overline{IA} \cap R \subseteq \overline{I}$ .

2.+3.: Assume  $d_J = \text{gcd}(J)$ ; then clearly  $J \subseteq Aa \iff a \mid d_J \iff d_J A \subseteq aA$ . Moreover,  $J \subseteq d_J A$ , so  $\overline{J} \subseteq d_J A \subseteq \bigcap_{J \subseteq Aa} Aa = \overline{J}$ . The opposite implications are obvious.

4. and 5.:  $\overline{aJ} = \bigcap_{Ja \subseteq Ac} Ac \subseteq \bigcap_{Ja \subseteq Aba} Aba = (\bigcap_{J \subseteq Ab} Ab) \cdot a = \overline{J} \cdot a$ . If  $J$  is divisorial we get  $\overline{aJ} \subseteq \overline{J} \cdot a = aJ \subseteq \overline{aJ}$ . Let  $g := \text{gcd}(J)$  and  $d := \text{gcd}(aJ)$ . Then  $ag$  is a common divisor of  $aJ$ , hence  $ag$  divides  $d$  and therefore  $d/a \in A$  is a common divisor of  $J$ . It follows that  $d/a$  divides  $g$ , hence  $d$  divides  $ag$ . So  $ag \mid d \mid ag \sim d$ . Now we get  $\overline{aJ} = \text{gcd}(aJ)A = a \text{gcd}(J)A = a\overline{J}$ .

6.: For every  $k \in K$  we have  $\overline{Jk} = \overline{Jk} \subseteq \overline{JK}$ , hence  $\overline{JK} \subseteq \overline{Jk}$  and  $\overline{J} \cdot \overline{K} \subseteq \overline{J \cdot K}$ . Clearly  $JK \subseteq \overline{JK}$ , hence  $\overline{JK} \subseteq \overline{J} \cdot \overline{K}$ .  $\square$

Let  $B$  be an arbitrary commutative ring and  $N \in B - \text{mod}$  a finitely generated  $B$ -module. Then  $N$  is torsion free of rank one  $\iff$  there is an ideal  $I \subseteq B$  containing a non zero-divisor, such that  $N \cong I \subseteq B$  are isomorphic as  $B$ -modules.

From now on let  $A$  be a normal noetherian domain, then  $A$  is a Krull-domain. Moreover for every finitely generated module  $M \in A - \text{mod}$  the following hold:

- (1)  $M^* := \text{Hom}_A(M, A) \cong \bigcap_{\mathfrak{p} \in \text{Spec}_1(A)} M_{\mathfrak{p}}^* \subseteq \mathbb{L} \otimes_A M^*$ .
- (2) If  $M$  is torsion free, then the canonical map  $c : M \rightarrow M^{**}$  induces an isomorphism

$$M^{**} \cong \bigcap_{\mathfrak{p} \in \text{Spec}_1(A)} M_{\mathfrak{p}}.$$

- (3) The fractional ideal  $I \in \mathcal{F}(A)$  is divisorial if and only if  $I$  is a reflexive  $A$ -module.
- (4)  $\ker(c) = \text{Tor}(M)$ , the torsion submodule of  $M$ , and  $M^*$  is reflexive.
- (5) For  $M, N \in A - \text{mod}$  one has

$$(\text{Hom}_A(M, N))^{**} \cong \text{Hom}_A(M^{**}, N^{**}).$$

**PROPOSITION 2.2.** *Let  $A$  be a normal noetherian domain, then there is a bijection between the divisor class group  $\mathcal{C}_A$  and the set of isomorphism classes of finitely generated reflexive  $A$ -modules of rank one.*

**Proof:** If  $M, N \in A - \text{mod}$  are f.g. reflexive  $A$ -modules of rank one, then  $M \cong I$  and  $N \cong J$  with divisorial ideals  $I, J \triangleleft A$ , so we can assume that  $M = I, N = J$  are divisorial ideals. Let  $\theta : I \rightarrow J$  be an isomorphism, then for any  $i, i' \in I$ ,  $\theta(ii') = i\theta(i') = i'\theta(i)$ , so  $\ell := \theta(i)/i \in \mathbb{L}$  with  $\ell \cdot I \subseteq J$ . By symmetry we have  $\ell^{-1} = i/\theta(i) = \theta^{-1}(\theta(i))/\theta(i) = \theta^{-1}(j)/j$  for every  $j \in J$ , hence  $j = \theta^{-1}(j)\ell$  and  $J \subseteq \ell I$ , so  $J = \ell \cdot I$ . It follows that the classes  $\text{cl}(J) := [\text{div}(J)]$  and  $\text{cl}(I) \in \mathcal{C}_A$  coincide.

Now assume  $\text{cl}(J) = \text{cl}(I) \in \mathcal{C}_A$ , then  $\text{div}(I) = \text{div}(J) + \text{div}(\ell A)$  for some  $\ell = a/b \in \mathbb{L}$ , hence

$$\text{div}(Ib) = \text{div}(I) + \text{div}(bA) = \text{div}(J) + \text{div}(aA) = \text{div}(Ja),$$

and replacing  $I$  by  $Ib \cong I$  and  $J$  by  $Ja \cong J$ , we can assume that  $\text{div}(I) = \text{div}(J)$ . Hence  $I_{\mathfrak{p}} = J_{\mathfrak{p}}$  for all  $\mathfrak{p} \in \text{Spec}_1(A)$ , so  $I \cong J$ , since these are reflexive  $A$ -modules.  $\square$

### 3. Relative Invariants

Now let  $G \subseteq \text{Aut}(A)$  be a finite group of ring automorphisms with corresponding ring of invariants  $R := A^G$  and quotient field  $\mathbb{K} = \mathbb{L}^G$ .

The Galois group  $G = \text{Gal}(\mathbb{L} : \mathbb{L}^G)$  acts as permutation group on  $\text{Spec}_1(A)$  and on the divisor group  $\mathcal{D}_A$  and there is an inclusion homomorphism  $\rho : \mathcal{D}_{A^G} \rightarrow \mathcal{D}_A$  satisfying

$$d(\mathfrak{q}) \mapsto e_{\mathfrak{q}} \cdot \left( \sum_{\mathfrak{Q} \in \text{Spec}_1(A) : \mathfrak{Q} \cap A^G = \mathfrak{q}} d(\mathfrak{Q}) \right) \in (\mathcal{D}_A)^G,$$

because the ramification index  $e_{\mathfrak{q}, A} := e_{\mathfrak{q}} := e(\mathfrak{Q}|\mathfrak{q})$  is constant for all  $\mathfrak{Q} \in \text{Spec}_1(A)$  over  $\mathfrak{q}$ . The group of invariants  $(\mathcal{D}_A)^G$  is a free abelian group with basis consisting of orbit sums

$$d(\mathfrak{Q})^+ := \sum_{g \in G/G_{\{\mathfrak{Q}\}}} d(g\mathfrak{Q}), \quad \mathfrak{Q} \in \text{Spec}_1(A).$$

Here  $G_{\{\mathfrak{Q}\}} := \text{Stab}_G(\mathfrak{Q})$  is the stabilizer (i.e. the decomposition group) of  $\mathfrak{Q}$ . Let  $\mathfrak{C}$  denote a fixed set of representatives for the  $G$ -orbits on  $\text{Spec}_1(A)$ , i.e.

$$\mathfrak{C} \cong \text{Spec}_1(A)/G \cong \text{Spec}_1(A^G).$$

Then we have a short exact sequence of abelian groups:

$$0 \longrightarrow \mathcal{D}_{A^G} \xrightarrow{\rho} (\mathcal{D}_A)^G \longrightarrow \bigoplus_{Q \in \mathfrak{C}} \mathbb{Z}/e_Q \mathbb{Z} \longrightarrow 0 \quad (3.1)$$

If  $aA \in (\mathcal{H}_A)^G$ , then  $g(a) = c_g a$  with  $c_g \in U(A)$  and  $gh(a) = c_{gh} a = g(c_h a) = g(c_h) c_g a$ , hence  $c_{gh} = c_g \cdot g(c_h)$ , so  $\lambda := c_{(\cdot)} \in Z^1(G, U(A))$  and  $a \in A_\lambda$ .

**LEMMA 3.1.** *Let  $\chi \in Z^1(G, U(A))$ , then  $0 \neq A_\chi$  is a reflexive  $R$ -module of rank one and is isomorphic to a divisorial ideal of  $R$ . The following hold:*

- (i) *for every  $0 \neq a \in A_{\chi^{-1}}$ ,  $\overline{aA_\chi A} \cap R = \overline{A_\chi a} = aA_\chi \triangleleft R$  is divisorial.*
- (ii) *Let  $\lambda \in B(G, U(A))$ , i.e.  $\lambda(g) = u^{-1}g(u)$  with  $u \in U(A)$ , and  $\mu := \chi \cdot \lambda$ . Then  $u \cdot A_\chi = A_\mu$  and  $A_\mu A = A_\chi A$ , which only depends on the class  $[\chi] \in H^1(G, U(A))$ .*
- (iii) *Assume  $A$  to be a normal domain. Then for every  $Q \in \text{Spec}_1(A)$ ,  $\nu_Q(\overline{A_\chi A}) < e(Q|q)$ .*

**Proof:** see [6] Lemmas 2.1/2.2.  $\circ$

Let  $Z_A^1(G, U(A)) := \{\lambda \in Z^1(G, U(A)) \mid A_\lambda \not\subseteq Q \ \forall Q \in \text{Spec}_1(A)\}$ . If  $\lambda, \mu \in Z^1(G, U(A))$  and  $Q \in \text{Spec}_1(A)$ , then  $A_{\lambda \cdot \mu} \subseteq Q$  implies  $A_\lambda \cdot A_\mu \subseteq A_{\lambda \cdot \mu} \subseteq Q$ , hence  $A_\lambda \subseteq Q$ , or  $A_\mu \subseteq Q$ . In other words,  $Z_A^1(G, U(A))$  is a subgroup of  $Z^1(G, U(A))$ , containing  $B(G, U(A))$  (since  $A_\lambda A = A$  for  $\lambda \in B(G, U(A))$ ). Therefore one can define

**DEFINITION 4.**  $H_A^1(G, U(A)) := Z_A^1(G, U(A))/B(G, U(A))$ .

**LEMMA 3.2.** *The sequence*

$$0 \longrightarrow H_A^1(G, U(A)) \longrightarrow H^1(G, U(A)) \xrightarrow{\Psi} \bigoplus_{Q \in \mathfrak{C}} \mathbb{Z}/e_Q \mathbb{Z}$$

with  $\Psi : [\chi] \mapsto (v_Q(\overline{A_\chi A}))_{Q \in \mathfrak{C}}$  is an exact sequence of abelian groups.

**Proof:** see [6] Lemmas 2.3.  $\circ$

The map

$$\mathcal{C}_{A^G} \rightarrow (\mathcal{D}_A)^G/(\mathcal{H}_A)^G \hookrightarrow (\mathcal{D}_A/\mathcal{H}_A)^G = (\mathcal{C}_A)^G$$

is essentially the natural map  $\phi : \mathcal{C}_{A^G} \rightarrow \mathcal{C}_A$  and we obtain

**COROLLARY 3.3.** *The kernel  $\ker(\phi)$  is naturally isomorphic to  $H_A^1(G, U(A)) \cong \ker(\Psi)$ . Moreover,  $\phi$  is injective if and only if the  $A_\chi$  are free  $R$ -modules for all  $\chi \in Z_A^1(G, U(A))$ .*

**Proof:** see [6] Lemma 2.4.  $\circ$

### 3.1. From now on we assume that $A$ is a factorial domain.

**DEFINITION 5.**

- (i) Let  $I_Q := G_{k(Q)} = \{g \in G \mid ga - a \in Q \ \forall a \in A\}$ , the inertia group of  $Q \in \text{Spec}_1(A)$ .
- (ii) An element  $g \in G$  is called a *reflection* on  $A$ , if  $g \in I_Q$  for some  $Q \in \text{Spec}_1(A)$ . The group

$$W := W_A := W_A(G) := \langle I_Q \mid Q \in \text{Spec}_1(A) \rangle$$

is a normal subgroup (since  $G$  acts on  $\text{Spec}_1(A)$ ) and is called the *subgroup of (generalized) reflections on  $A$* .

**THEOREM 3.4.** *Let  $A$  and  $G$  be as above and assume that  $A$  is a factorial domain. Then  $\mathcal{C}_{A^G} \cong$*

$$H_A^1(G, U(A)) \cong \tilde{H} := \{\rho \in H^1(G, U(A)) \mid \text{res}_{I_Q}(\rho) = 1 \text{ in } H^1(I_Q, U(A_Q)), \forall Q \in \text{Spec}_1(A)\}.$$

*In explicit form: Let  $I$  be a divisorial ideal of  $A^G$ , then  $\overline{IA} = aA$  with semi-invariant  $a \in A$ . If  $\theta_a \in Z^1(A, U(A))$  is the corresponding cocycle, i.e.  $g(a) = \theta_a(g)a$  for every  $g \in G$ , the class  $[I] \in \mathcal{C}_{A^G}$  corresponds to the element  $[\theta_a] \in \tilde{H}$ .*

**Proof:** See [6] Lemma 2.4. The explicit form can be seen by following up the isomorphism described there.  $\circ$

**PROPOSITION 3.5.** *For  $\chi \in Z^1(G, U(A))$  the following hold:*

- (i)  $\overline{A_\chi A} = d_\chi A$ ,  $d_\chi := \gcd(A_\chi) \in A_{\mu_\chi}$  with  $\mu_\chi \in Z^1(G, U(A))$  and a uniquely defined element  $[\mu_\chi] \in H^1(G, U(A))$ .
- (ii)  $A_\chi$  defines a unique class  $\text{cl}(A_\chi) \in \mathcal{C}_R$ , which satisfies  $\text{cl}(A_\chi) = [\chi^{-1}\mu_\chi] \in \tilde{H}$  (see 3.4).
- (iii)  $A_\chi$  is a free  $R$ -module if and only if  $[\chi] = [\mu_\chi] \in H^1(G, U(A))$ .

**Proof:** (i): This follows from 2.1.

(ii): For every  $a \in A_{\chi^{-1}}$  the ideal  $aA_\chi \triangleleft R$  is divisorial and we get from 2.1:  $\overline{aA_\chi A} = ad_\chi A$  with  $ad_\chi \in A_{\chi^{-1}\mu_\chi}$ . Hence  $\text{cl}(A_\chi) = [\chi^{-1}\mu_\chi] \in \tilde{H}$  by 3.4.

(iii): This follows immediately from the above.  $\circ$

**LEMMA 3.6.** *For  $[\chi] \in H^1(G, U(A))$  the following are equivalent:*

- (i)  $[\chi] \in \tilde{H} = H_A^1(G, U(A))$ ;
- (ii)  $d_\chi \in U(A)$ ;
- (iii)  $[\chi^{-1}] = \text{cl}(A_\chi) \in \mathcal{C}_R \cong \tilde{H}$ ;
- (iv)  $\overline{A_\chi A} = A$ .

**Proof:** “(i)  $\iff$  (ii)”: Let  $[\chi] \in \tilde{H}$ , then there is a divisorial ideal  $J \trianglelefteq R$  with  $\text{cl}(J) = [\chi^{-1}]$ , i.e.  $\overline{JA} = fA$  with  $f \in A_{\chi^{-1}}$ . The divisorial ideal  $I := fA_\chi \triangleleft R$  satisfies

$$\overline{fA_\chi A} = \overline{IA} = f \cdot \overline{A_\chi A} = fd_\chi A.$$

Hence  $J = \overline{JA} \cap R = fA \cap R = fA_\chi = I$ , so  $fd_\chi A = \overline{IA} = \overline{JA} = fA$  and  $d_\chi \in U(A)$ . On the other hand, if  $d_\chi \in U(A)$ , then  $[\mu_\chi] = 1 \in H^1(G, U(A))$  and  $[\chi^{-1}] = [\chi^{-1}][\mu_\chi] \in \tilde{H}$ .

“(i)  $\iff$  (iii)” and “(ii)  $\iff$  (iv)” follow from 3.5.  $\circ$

**COROLLARY 3.7.** *For  $\chi \in Z^1(G, U(A))$  we have  $A_\chi = d_\chi \cdot A_{\chi\mu_\chi^{-1}}$ . Assume  $A_\chi = a \cdot S$  with  $S \subseteq A$  and  $a \in A$ . Then  $a \mid d_\chi$  and the following hold:*

- (i)  $a \sim d_\chi \iff S = A_\lambda$  with  $[\lambda] = [\chi\mu_\chi^{-1}] \in \tilde{H}$  (i.e.  $A_\lambda \cong A_1 = R$  in  $R\text{-mod.}$ )
- (ii)  $1_A \in S \iff S = R \iff d_\chi \sim a \in A_\chi$ .

**Proof:** Since  $d_\chi = \gcd(A_\chi)$ ,  $A_\chi d_\chi^{-1} \subseteq A_{\chi\mu_\chi^{-1}}$ , hence  $d_\chi \cdot A_{\chi\mu_\chi^{-1}} \subseteq A_\chi \subseteq d_\chi \cdot A_{\chi\mu_\chi^{-1}}$ , so

$$A_\chi = d_\chi \cdot A_{\chi\mu_\chi^{-1}}.$$

(i): If  $A_\chi = aS$  with  $S \subseteq A \ni a$ , then clearly  $a \mid d_\chi$ . If  $a = ud_\chi$  with  $u \in U(A)$ , then  $d_\chi \cdot A_{\chi\mu_\chi^{-1}} = A_\chi = ud_\chi S$ , hence  $S = u^{-1}A_{\chi\mu_\chi^{-1}} = A_\lambda$  with  $[\lambda] = [\chi\mu_\chi^{-1}]$ .

Assume  $S = A_\lambda \cong R$ , then  $d_\chi A = \overline{A_\chi A} = \overline{aA_\lambda A} = \overline{aA_\lambda A} = aA$  by 3.6; hence  $a \sim d_\chi$ .

(ii): If  $1_A \in S$ , then  $a \in A_\chi$ , therefore  $d_\chi \mid a$  and  $S = 1/aA_\chi \subseteq R$ . Hence  $A_\chi \subseteq aR \subseteq A_\chi$  and  $R = 1/aA_\chi = S$ .

If  $S = R$ , then  $A_\chi = aR$ , so  $\gcd(A_\chi) \ni a \in A_\chi$ .

If  $d_\chi \sim a \in A_\chi$ ,  $aR \subseteq A_\chi$ , hence  $1_A \in R \subseteq 1/aA_\chi = S$ .  $\circ$

**COROLLARY 3.8.** Let  $[\lambda] \in H^1(G, U(A))$  such that  $A_\lambda = dR$ . Then for every  $[\sigma] \in \tilde{H}$  we have

$$d = \gcd(A_{\lambda\sigma}) \sim d_{\lambda\sigma},$$

i.e.  $d$  and  $d_{\lambda\sigma}$  are associated. In particular  $d = d_\lambda \cdot u$  with  $u \in U(R)$  and  $A_\lambda = Ad_\lambda$ .

**Proof:** We have  $dA = \overline{dA_\sigma A} = \overline{dRA_\sigma A} = \overline{A_\lambda A_\sigma A} \subseteq \overline{A_{\lambda\sigma} A} = d_{\lambda\sigma} A =$

$$d_{\lambda\sigma} \overline{A_{\sigma^{-1}} A} = \overline{d_{\lambda\sigma} A_{\sigma^{-1}} A} = \overline{d_{\lambda\sigma} AA_{\sigma^{-1}} A} = \overline{A_{\lambda\sigma} AA_{\sigma^{-1}} A} = \overline{A_{\lambda\sigma} AA_{\sigma^{-1}} A} \subseteq \overline{A_\lambda A} = dA.$$

It follows that  $d_\lambda = u \cdot d$  with  $u \in U(A) \cap R = U(R)$ .  $\circ$

**COROLLARY 3.9.** Let  $[\lambda] \in H^1(G, U(A))$  such that  $A_\lambda = dR$ . Then for every  $[\sigma] \in \tilde{H}$  we have

$$d = \gcd(A_{\lambda\sigma}) \sim d_{\lambda\sigma},$$

i.e.  $d$  and  $d_{\lambda\sigma}$  are associated.

**Proof:** We have  $dA = \overline{dA_\sigma A} = \overline{dRA_\sigma A} = \overline{A_\lambda A_\sigma A} \subseteq \overline{A_{\lambda\sigma} A} = d_{\lambda\sigma} A = d_{\lambda\sigma} \overline{A_{\sigma^{-1}} A} =$

$$\overline{d_{\lambda\sigma} A_{\sigma^{-1}} A} = \overline{d_{\lambda\sigma} AA_{\sigma^{-1}} A} = \overline{A_{\lambda\sigma} AA_{\sigma^{-1}} A} = \overline{A_{\lambda\sigma} AA_{\sigma^{-1}} A} \subseteq \overline{A_\lambda A} = dA.$$

$\circ$

**THEOREM 3.10.** Let  $\mathcal{P}_{G,A} := \{[\lambda] \in H^1(G, U(A)) \mid A_\lambda = d_\lambda R\}$ . Then

$$\mathcal{P}_{G,A} = \{[\lambda] \in H^1(G, U(A)) \mid \text{cl}(A_\lambda) = 1\}, \quad \mathcal{P}_{G,A} \cap \tilde{H} = 1 \text{ and } H^1(G, U(A)) = \uplus_{[\lambda] \in \mathcal{P}_{G,A}} \tilde{H} \cdot [\lambda].$$

So  $\mathcal{P}_{G,A} \subseteq H^1(G, U(A))$  is a transversal of the cosets of the subgroup  $\tilde{H} \subseteq H^1(G, U(A))$ .

For every  $[\chi] \in H^1(G, U(A))$  let  $[\mu_\chi] \in H^1(G, U(A))$  be the character of  $d_\chi := \gcd(A_\chi)$ , i.e.  $d_\chi \in A_{\mu_\chi}$ . Then the following hold

(i)  $\text{cl}(A_\chi) = [\chi^{-1}][\mu_\chi]$  with  $\{[\mu_\chi]\} = \mathcal{P}_{G,A} \cap \tilde{H} \cdot [\chi]$ .

(ii) The map

$$\mu : H^1(G, U(A)) \rightarrow \mathcal{P}_{G,A}, \quad [\chi] \mapsto [\mu_\chi]$$

satisfies  $\mu \circ \mu = \mu$  and it is a projection operator onto the distinguished transversal  $\mathcal{P}_{G,A}$ .



**Proof:** The equation  $\mathcal{P}_{G,A} \cap \tilde{H} = 1$  follows from 3.6 (iv).

Let  $[\lambda], [\delta] \in \mathcal{P}_{G,A}$  with  $[\sigma] := [\lambda]^{-1}[\delta] \in \tilde{H}$ , then  $[\delta] = [\lambda][\sigma]$ , hence by Corollary 3.9,  $d_\delta \sim d_\lambda$  and  $[\lambda] = [\delta]$ . This shows that every  $\tilde{H}$  coset contains at most one element in  $\mathcal{P}_{G,A}$ . Let  $[\chi] \in H^1(G, U(A))$ , then

$$\overline{A_\chi A} = d_\chi A \subseteq A_{\mu_\chi^{(1)}} A \subseteq \overline{A_{\mu_\chi^{(1)}} A} = d_{\mu_\chi^{(1)}} A \subseteq \overline{A_{\mu_\chi^{(2)}} A} = d_{\mu_\chi^{(2)}} A \subseteq \overline{A_{\mu_\chi^{(3)}} A} = \dots$$

with

$$[\chi] \equiv [\mu_\chi^{(1)}] \equiv [\mu_\chi^{(2)}] \equiv \dots \pmod{\tilde{H}}.$$

It is clear that this ascending chain of divisorial ideals must be stationary, hence we will eventually have

$$d_{\mu_\chi^{(i)}} A = d_{\mu_\chi^{(i+1)}} A = d_{\mu_\chi^{(\infty)}}, \text{ and therefore } [\mu_\chi^{(i)}] = [\mu_\chi^{(i+1)}] = [\mu_\chi^{(\infty)}] =: [\lambda]$$

with

$$\overline{A_\lambda A} = d_\lambda A \subseteq A_\lambda A \subseteq \overline{A_\lambda A} \text{ and } [\chi] \equiv [\mu_\chi] \equiv \dots \equiv [\mu_\chi^{(\infty)}] = [\lambda] \pmod{\tilde{H}}.$$

It follows that  $d_\lambda = \gcd(A_\lambda) \in A_\lambda$ , hence  $A_\lambda = d_\lambda R$ , so  $[\lambda] \in \mathcal{P}_{G,A} \cap \tilde{H} \cdot \chi$ .

It now follows from Corollary 3.9 that

$$d_\lambda \sim d_{\mu_\chi^{(i)}} \sim d_{\mu_\chi^{(i-1)}} \sim d_{\mu_\chi^{(i-2)}} \sim \dots \sim d_{\mu_\chi^{(1)}} \sim d_\chi.$$

So  $[\mu_\chi] := [\mu_\chi^{(1)}] \in \mathcal{P}_{G,A} \cap \tilde{H} \cdot [\chi]$ . By construction we have  $d_{\mu_\chi} \sim d_\chi$ , hence  $\mu \circ \mu([\chi]) = \mu([\chi])$ , which finishes the proof.  $\circ$

**COROLLARY 3.11.** *For every  $[\lambda] \in \mathcal{P}_{G,A}$  we have  $\mathcal{C}_R = \{\text{cl}(A_\chi) \mid \chi \in \tilde{H} \cdot [\lambda]\}$ , i.e. if  $\chi$  ranges through the full coset  $\tilde{H} \cdot [\lambda]$ , then the  $A_\chi$  form a transversal of all isomorphism types of rank one reflexive  $R$ -modules.*

*Alternatively the set  $\{A_{\chi\mu_\chi^{-1}} \mid \chi \in Z^1(G, U(A))\}$  is also a full set of representatives of reflexive rank one  $R$ -modules.*

**Proof:** Every rank one reflexive  $R$ -module is isomorphic to a divisorial ideal of  $R$ , the isomorphism type of which is uniquely determined by its ideal class. From Corollary 3.10 we see that  $[\mu_{\sigma\lambda}] = \text{eigencharacter of } (d_{\sigma\lambda}) = \text{eigencharacter of } (d_\lambda) = [\lambda]$ , hence we get

$$\text{cl}(A_{\sigma\lambda}) = [\sigma]^{-1}[\lambda]^{-1}[\mu_{\sigma\lambda}] = [\sigma]^{-1}[\lambda]^{-1}[\lambda] = [\sigma]^{-1}.$$

The last statement follows from 3.7, since  $A_\chi = d_\chi \cdot A_{\chi\mu_\chi^{-1}} \cong A_{\chi\mu_\chi^{-1}}$  in  $R\text{-mod.}$   $\circ$

### 3.2. $A$ noetherian, factorial domain, $U(A) = U(k)$

**From now on we assume that  $A$  is a noetherian factorial domain with  $U(A) = U(k)$  with  $k \subseteq A$ , a field of characteristic  $p \geq 0$ .**

Let  $P = a_P A \in \text{Spec}_1(A)$  and  $\sigma \in I := I_P$ . Then for  $u \in k$ ,  $(\sigma - 1)(u) \in k \cap P = 0$ , so  $\sigma(u) = u$  and  $W \subseteq \text{Aut}_k(A)$ . Clearly  $P$  is  $I$ -stable, so  $\sigma(a_P) = \delta_P(g)a_P$  and the map

$$\delta_P : I_P \rightarrow U(k), \sigma \mapsto \delta_P(g) = a_P^{-1}\sigma(a_P)$$

is an element in  $Z^1(I, U(k)) = \text{Hom}(I, U(k))$ .

**LEMMA 3.12.** *For  $P \in \text{Spec}_1(A)$ ,  $I := I_P$  and  $e := e(P|P \cap R)$  we have  $\text{Hom}(I, U(k)) = \text{Hom}(I, U(A_P)) = \langle \delta_P \rangle \cong \mathbb{Z}/e\mathbb{Z}$ . There is a short exact sequence*

$$0 \longrightarrow \mathcal{C}_{A^\sigma} \longrightarrow H^1(G, U(k)) \longrightarrow \bigoplus_{Q \in \mathfrak{C}} \text{Hom}(I_Q, U(k)) \longrightarrow 0$$

In particular  $\mathcal{C}_{A^G} \cong H^1(G/W, U(k))$ .

**Proof:** see [6] Lemmas 2.6. In addition to this, we only need to show that  $\tilde{H} = H^1(G/W, U(k))$ . Let  $[\chi] \in \tilde{H}$  with  $\chi \in Z^1(G, U(k))$ , then for  $g, h \in W$ ,  $\chi(gh) = \chi(g)g(\chi(h)) = \chi(g)\chi(h)$ , since  $W$  acts trivially on  $k$ . Moreover  $g$  and  $h$  are products of elements on which  $\chi$  is 1, hence  $\chi|_W = 1$ . We view  $Z^1(G/W, U(k))$  as a subset of  $Z^1(G, U(k))$  in a natural way. Then, again since  $W$  acts trivially on  $k$  we have  $B^1(G, U(k)) \subseteq Z^1(G/W, U(k))$ , hence  $B^1(G, U(k)) = B^1(G/W, U(k))$ , so  $\mathcal{C}_{A^G} = \tilde{H} \cong Z^1(G/W, U(k))/B^1(G/W, U(k)) = H^1(G/W, U(k))$ .  $\circ$

### 3.3. A noetherian, factorial domain, $U(A) = U(k)$ with trivial $G$ -action

Then  $H^1(G, U(A)) = G^* := \text{Hom}(G, U(k))$ , the group of linear  $k$ -characters of  $G$ . If  $N \trianglelefteq G$  is a normal subgroup, then the restriction map yields a short exact sequence

$$1 \rightarrow (G/N)^* \rightarrow G^* \rightarrow N^* \rightarrow 1.$$

**COROLLARY 3.13.** *There is an isomorphism  $\text{ch} : \mathcal{C}_{A^G} \cong \overline{G}^* = \ker(\text{res}|_W)$ , where  $\overline{G} := G/W$  and  $\text{res}|_W : G^* \rightarrow W^*$  is the restriction map on characters.*

## 4. Quasi-Gorenstein Rings of Invariants

Now let  $k$  be a field and  $A$  a finitely generated normal  $k$ -algebra with  $U(A) = U(k)$ , such that the quotient field  $\mathbb{L} := \text{Quot}(A)$  is separable over  $k$ . Let  $G \subseteq \text{Aut}(A)$  be a finite group with ring of invariants  $R := A^G$ . Then  $k$  is a separable algebraic extension of  $k' := k^G \subseteq \mathbb{K} := \text{Quot}(R)$  and  $\mathbb{L}$  as well as  $\mathbb{K}$  are separable over  $k'$ . By Noether-normalization there is a  $k'$ -polynomial ring  $\mathcal{F} \subseteq R := A^G$  such that  $\mathcal{F}R$  and  $\mathcal{F}A$  are finitely generated modules, i.e.  $\mathcal{F} = k'[f_1, \dots, f_d]$ , with  $(f_1, \dots, f_d)$  a system of parameters of  $R$  as  $k'$ -algebra. It follows from [5] Cor. 16.18 pg.403, that  $\mathcal{F}$  can be chosen such that  $\mathbb{L}$  and  $\mathbb{K}$  are separable over  $\text{Quot}(\mathcal{F})$ . For technical reasons, which become clear later in section 6, we choose and fix  $\mathcal{F}$  in such a way.

**PROPOSITION 4.1.** *Let  $\mathcal{P} \subseteq A$  as above be quasi-Gorenstein. Then for every  $\mathbf{Q} \in \text{Spec}(A)$ , the localisation  $A_{\mathbf{Q}}$  is Cohen-Macaulay if and only if  $A_{\mathbf{Q}}$  is Gorenstein. In other words, the Cohen-Macaulay and Gorenstein loci of  $A$  coincide.*

**Proof:** Let  $\mathbf{Q} \in \text{Spec}(A)$  be such that  $A_{\mathbf{Q}}$  is Cohen-Macaulay. Set  $\mathbf{q} = \mathbf{Q} \cap \mathcal{P} \in \text{Spec}(\mathcal{P})$  and let  $\mathbf{Q} := \mathbf{Q}_1, \dots, \mathbf{Q}_k$  be the primes of  $A$  lying over  $\mathbf{q}$ . Since  $\text{Hom}_{\mathcal{P}}(A, \mathcal{P}) \cong A$  and  $\widehat{A}_{\mathbf{q}} \cong \times_{i=1}^k \widehat{A}_{\mathbf{Q}_i}$ , we get  $\widehat{A}_{\mathbf{q}} \cong (\widehat{\text{Hom}_{\mathcal{P}}(A, \mathcal{P})})_{\mathbf{q}} \cong \text{Hom}_{\mathcal{P}}(A, \mathcal{P}) \otimes_{\mathcal{P}} \widehat{\mathcal{P}}_{\mathbf{q}} \cong \text{Hom}_{\widehat{\mathcal{P}}_{\mathbf{q}}}(\widehat{A}_{\mathbf{q}}, \widehat{\mathcal{P}}_{\mathbf{q}}) \cong \times_{i=1}^k \text{Hom}_{\widehat{\mathcal{P}}_{\mathbf{q}}}(\widehat{A}_{\mathbf{Q}_i}, \widehat{\mathcal{P}}_{\mathbf{q}}) \cong \times_{i=1}^k \widehat{A}_{\mathbf{Q}_i}$ . Let  $1_{\widehat{A}_{\mathbf{q}}} = \sum_{i=1}^k e_i$  with  $e_i e_j = \delta_{ij}$ , then  $\widehat{A}_{\mathbf{Q}_i} = e_i \widehat{A}_{\mathbf{q}} \cong e_i \text{Hom}_{\widehat{\mathcal{P}}_{\mathbf{q}}}(\widehat{A}_{\mathbf{q}}, \widehat{\mathcal{P}}_{\mathbf{q}}) \cong \text{Hom}_{\widehat{\mathcal{P}}_{\mathbf{q}}}(e_i \widehat{A}_{\mathbf{q}}, \widehat{\mathcal{P}}_{\mathbf{q}}) \cong \text{Hom}_{\widehat{\mathcal{P}}_{\mathbf{q}}}(\widehat{A}_{\mathbf{Q}_i}, \widehat{\mathcal{P}}_{\mathbf{q}})$ . Since  $A_{\mathbf{Q}_1}$  is Cohen-Macaulay, so is  $\widehat{A}_{\mathbf{Q}_1}$  and it is finite over  $\widehat{\mathcal{P}}_{\mathbf{q}}$ . It follows that  $\text{Hom}_{\widehat{\mathcal{P}}_{\mathbf{q}}}(\widehat{A}_{\mathbf{Q}_i}, \widehat{\mathcal{P}}_{\mathbf{q}})$  is the unique canonical module  $\omega_{\widehat{A}_{\mathbf{Q}_i}}$  (up to isomorphism) of  $\widehat{A}_{\mathbf{Q}_i}$ . Therefore  $\omega_{\widehat{A}_{\mathbf{Q}_i}} \cong \widehat{A}_{\mathbf{Q}_i}$ . It is generally true, that for a finitely generated  $A_{\mathbf{Q}}$ -module  $M$ , the completion  $M \otimes_{A_{\mathbf{Q}}} \widehat{A}_{\mathbf{Q}}$  is canonical for  $\widehat{A}_{\mathbf{Q}}$ , if and only if  $M$  is canonical for  $A_{\mathbf{Q}}$ , so we conclude that  $\omega_{A_{\mathbf{Q}}} \cong A_{\mathbf{Q}}$  and  $A_{\mathbf{Q}}$  is Gorenstein.  $\circ$

DEFINITION 6. For a normal subring  $S \subseteq A$  such that  $S \hookrightarrow A$  is finite and  $\text{Quot}(A)$  is separable over  $\text{Quot}(S)$  let  $\mathcal{D}_{A,S} \trianglelefteq A$  denote the corresponding Dedekind different.

It is well known that  $\mathcal{D}_{A,S}$  and its inverse  $\mathcal{D}_{A,S}^{-1}$  are divisorial (fractional) ideals with

$$\mathcal{D}_{A,S}^{-1} \cong \text{Hom}_S(A, S).$$

**Now we assume in addition that  $A$  is a factorial domain (see subsection 3.2).**

Let  $S := A^W$ , then by 3.12  $S$  is also factorial. The following lemma is well known (at least in the context of Dedekind domains appearing in number theory):

LEMMA 4.2. For any  $W \subseteq H \subseteq G$  the following holds:

- (i)  $\mathcal{D}_{A,A^G} = \mathcal{D}_{A,A^H}$ .
- (ii)  $\mathcal{D}_{A^H,A^G} = (1) = A^H$ .

In particular the extension  $A^G \hookrightarrow A^W$  is unramified in height one.

Using the fact that  ${}_R S$  is unramified in height one we can now prove the main result:

**Proof** (of Theorem 1.2): We have  $\mathcal{D}_{S,R} = S$ ,  $R^* = \mathcal{D}_{R,\mathcal{F}}^{-1}$  and  $\mathcal{D}_{S,\mathcal{F}} = Sd$ , a principal ideal, since  $S$  is a factorial domain. It follows that  $S^* = S\theta_S$ , where  $\theta_S \in S^*$  can be identified with an element in  $\text{Quot}(S)$ . Since the fractional ideal  $\mathcal{D}_{S,\mathcal{F}}^{-1}$  is  $G/W$ -stable  $\theta_S$  is a relative invariant with character  $\chi_S \in \tilde{H}$ . By the Dedekind-tower theorem,  $\mathcal{D}_{S,\mathcal{F}} = \overline{\mathcal{D}_{S,R}\mathcal{D}_{R,\mathcal{F}}} \trianglelefteq S$ , which implies (first locally at height one primes, then globally):

$$S^* \cong S\theta_S = \mathcal{D}_{S,\mathcal{F}}^{-1} = \overline{\mathcal{D}_{S,R}^{-1}S\mathcal{D}_{R,\mathcal{F}}^{-1}} = \overline{S\mathcal{D}_{R,\mathcal{F}}^{-1}} \subseteq \text{Quot}(S).$$

There is a suitable element  $r \in R$  with  $rS\theta_S \subseteq S$  and therefore  $rS\theta_S = \overline{rS\mathcal{D}_{R,\mathcal{F}}^{-1}} \subseteq S$ . Hence we get  $\overline{rS\mathcal{D}_{R,\mathcal{F}}^{-1}} \cap R = r\mathcal{D}_{R,\mathcal{F}}^{-1} = rS\theta_S \cap R$ , so  $R^* \cong \mathcal{D}_{R,\mathcal{F}}^{-1} = S\theta_S \cap \text{Quot}(R) = S_{\chi_S^{-1}} = A_{\chi_S^{-1}}$ , where the isomorphism is one of  $R$ -modules. Since  $\chi_S \in \tilde{H}$  we have  $[\mu_{\chi_S}] = 1$ , so  $\text{ch}(\text{cl}(A_{\chi_S^{-1}})) = [\chi_S]$ . The equation  $\chi_S = \chi_A \cdot \chi_{A,S}^{-1}$  follows immediately from

$$\overline{\mathcal{D}_{A,R}\mathcal{D}_{R,\mathcal{F}}} = \overline{\mathcal{D}_{A,S}\mathcal{D}_{S,\mathcal{F}}}$$

and  $\mathcal{D}_{A,S} = \mathcal{D}_{A,R}$ . The remaining statements follow immediately.  $\circ$

**Proof** of Corollary 1.4: Since  $\tilde{W}/W$  is generated by  $p$ -elements, it follows that  $\mathcal{C}_{\tilde{S}} = \text{Hom}(\tilde{W}/W, U(k)) = 1$ , hence  $\text{Hom}(G/W, U(k)) = \text{Hom}(G/\tilde{W}, U(k))$  and  $\tilde{S}$  is a factorial domain, hence quasi Gorenstein. Using Lemma 4.2 the remaining arguments are exactly as above with  $W$  replaced by  $\tilde{W}$  and  $S$  by  $\tilde{S}$ .  $\circ$

**Proof** of Corollary 1.3: this follows immediately from Theorem 1.2 and Proposition 4.1.  $\circ$

## 5. The graded connected case

The application of Theorem 1.2 depends on the determination of  $[\chi_S]$  or, equivalently  $[\chi_A]$  and  $[\chi_{A,S}]$ . If  $G$  acts trivially on  $k$ , then these are linear characters in  $\text{Hom}(G/W, U(k))$  or  $\text{Hom}(G, U(k))$ , respectively. In this section we investigate these characters in the case where  $A$  is a graded connected Cohen-Macaulay ring.

So throughout this section  $A = \sum_{i \geq 0} A_i$  is an  $\mathbb{N}_0$  graded connected noetherian normal  $k$ -algebra, i.e.  $A_0 = k$  with  $U(A) = U(k)$  and  $G \subseteq \text{Aut}_k(A)$  a finite group of graded  $k$ -algebra automorphisms. We will also assume that  $A$  is a Cohen-Macaulay domain, i.e.  $A$  is a free module over some (and then every) parameter algebra  $\mathcal{F} \subseteq A$ . We keep the previous notation,

so  $R = A^G \hookrightarrow A$  is a finite extension of noetherian normal domains. Let  $y_1, y_2, \dots, y_d \in R$  be a homogeneous system of parameters (hsop) with  $d_i := \deg(y_i)$ ,  $d = \dim(R) = \dim(A)$ , and set  $\mathcal{F} := k[y_1, \dots, y_d]$ .

**DEFINITION 7.** Let  $V := \oplus_{n \geq 0} V_n$  be an  $\mathbb{N}_0$  graded  $k$ -vectorspace and  $G$  a finite group acting on  $V$  by graded  $k$ -linear automorphisms. We define the (Brauer-) character series

$$H_{V,g}^{(Br)}(t) := \sum_{n=0}^{\infty} \chi_{V_n}(g) t^n,$$

where  $\chi_{V_n}$  is the (Brauer-) character afforded by the action of  $G$  on  $V_n$ . Note that  $H_{V,g}(t) \in k[[t]]$ , whereas  $H_{V,g}^{Br}(t) \in \mathbb{Q}(\epsilon)[[t]]$ , where  $\epsilon$  is a primitive  $\text{order}(g)$ -th root of unity in  $\mathbb{C}$ .

Note that

$$H_A(t) := H_{A,\text{id}}^{Br}(t) = \sum_{i \geq 0} \dim_k(A_i) t^i \in \mathbb{Q}(t)$$

is the ordinary Hilbert-series of  $A$ . Let

$$U := \overline{A} := A/\mathcal{F}^+A,$$

where  $\mathcal{F}^+ := (y_1, \dots, y_d) \trianglelefteq \mathcal{F}$  is the unique maximal homogeneous ideal of  $\mathcal{F}$ . Then  $\mathcal{F} \otimes_k U$  is the projective cover of  $\mathcal{F}A$  in  $\mathcal{F} \text{--} \text{mod}$ , hence, as  $\mathcal{F}A$  is free, we have  $\mathcal{F} \otimes_k U \cong A$  as  $\mathcal{F}$ -modules. Moreover

$$U = \oplus_{i=0}^{\beta} U_i = \oplus_{i=1}^{\ell} k\xi_i,$$

where we choose a homogeneous  $k$ -basis  $\{\xi_i \mid i = 1, \dots, \ell\}$  with  $\deg(\xi_i) =: \beta_i \leq \beta_{i+1}$ ,  $\beta := \beta_{\ell}$  and  $\ell := \dim_k(U)$ . We also will choose an  $\mathcal{F}$ -basis  $\mathcal{B} := \{s_i \mid i = 1, \dots, \ell\}$  of  $A$ , such that  $s_i + \mathcal{F}^+A = \overline{s_i} = \xi_i$  for  $i = 1, \dots, \ell$ .

Note that  $G$  acts on  $A$  and  $U$  and if  $g(\xi_i) = \sum_{j=1}^{\ell} g_{ji} \xi_j$  with  $(g_{ji}) \in k^{\ell \times \ell}$ , then

$$g(s_i) = \sum_{j=1}^{\ell} g_{ji} s_j + \mathcal{X}$$

with  $\mathcal{X} \in \mathcal{F}^+A$ . For each  $j$  let  $\tilde{A}_j := \langle \mathcal{B} \rangle_k \cap A_j$ , then  $A_i = \oplus_{m+n=i} \mathcal{F}_m \otimes_k \tilde{A}_n$  and it is easily seen that

$$\chi_{A_i}(g) = \sum_{m+n=i} \dim_k(\mathcal{F}_m) \cdot \chi_{U_n}(g) = \text{coeff}_i(H_{\mathcal{F}}^{Br}(t) \cdot H_{U,g}^{Br}(t)).$$

Hence

$$H_{A,g}^{Br}(t) = H_{\mathcal{F}}(t) \cdot H_{U,g}^{Br}(t).$$

Since  $H_{\mathcal{F}}(t) = \frac{1}{\prod_{i=1}^{\ell} (1-t^{d_i})}$  and  $H_{U,g}^{Br}(t) \in \mathbb{Q}(\epsilon)[t]$ , we get

**LEMMA 5.1.** *The Brauer-character series of  $A$  are rational, i.e.  $H_{A,g}^{Br}(t) \in \mathbb{Q}(\epsilon)(t)$ .*

**Now we assume in addition that  $A$  is Gorenstein.** It is then well known that

$$H_A(t) = (-1)^d t^{a(A)} H_A(1/t),$$

where  $a(A) = \deg(H_A(t))$  is the degree of  $H_A(t)$ . This symmetry is induced by the duality of the corresponding artinian Gorenstein algebra

$$U = \overline{A} := A/\mathcal{F}^+A,$$

where  $\mathcal{F}^+ := (y_1, \dots, y_d) \trianglelefteq \mathcal{F}$  is the unique maximal homogeneous ideal of  $\mathcal{F}$ . For later use we recall the details:

There is a graded embedding

$$U/U^+[-\beta] \hookrightarrow k[-\beta] \subseteq {}_U U_\beta, \quad k = ((U/U^+)[- \beta])_\beta \ni \lambda \mapsto \lambda \xi_\ell.$$

It follows from [4] that  ${}_U U$  is injective with  $\text{Soc}(U) \cong k$  (up to shift), hence

$$k[-\beta] \cong U/U^+[-\beta] \cong \text{Soc}({}_U U).$$

It is well known that  ${}^*E(k) \cong U^* := \text{Hom}_k(U, k)$ , where  ${}^*E(k)$  denotes the graded  ${}^*$ injective hull of  $k = U_0$  (see [4] for the definition of  ${}^*$ injectivity). Note that  $U = \bigoplus_{i=0}^\beta U_i$ ; choosing a homogeneous dual  $k$ -basis  $\{\xi_i^* \mid i = 1, \dots, \ell\}$  (such that  $\xi_i^*(\xi_j) = \delta_{i,j}$  and  $\deg(\xi_i^*) = -\deg(\xi_i)$ ), we see that  $U^* = \bigoplus_{i=0}^\beta (U^*)_{-i}$  with  $k \cong \text{Soc}(U^*) \cong U_0^*$  and  $\dim_k(U^*)_{-i} = \dim_k U_i$ . Since  ${}_U U$  is injective and indecomposable we conclude

$${}_U U \cong {}^*E(\text{Soc}({}_U U)) \cong {}^*E(k[-\beta]) \cong {}^*E(k)[- \beta] \cong U^*[-\beta].$$

It follows that  $\dim_k(U_i) = \dim_k(U^*[-\beta]_i) = \dim_k((U^*)_{i-\beta}) = \dim_k(U_{\beta-i})$ , hence  $H_U(t) = t^\beta H_U(1/t) = H_U^*(t)$ . Since  $\text{Rad}(U^*) = \text{Soc}(U)^\perp = \langle \xi_1^*, \dots, \xi_{\ell-1}^* \rangle$  we have  ${}_U U^* = U \cdot \xi_\ell^*$  as well as a non-degenerate associative bilinear form

$$\kappa(\cdot, \cdot) : U \times U \rightarrow k, \quad \kappa(\xi, \xi') = \xi_\ell^*(\xi \cdot \xi').$$

It follows from  $\text{Soc}(U) = k \cdot \xi_\ell$ , that for  $g \in G$ ,  $g(\xi_\ell) = \lambda(g)\xi_\ell$ , with some linear character  $\lambda \in \text{Hom}(G, U(k))$ . Since the  $G$ -action preserves degrees, we have  $g(\xi_j) \in \sum_{n < \beta} U_n$ , hence  $g^{-1}\xi_\ell^*(\xi_j) = \xi_\ell^*(g(\xi_j)) = 0$  for every  $j < \ell$  and

$$g^{-1}\xi_\ell^*(\xi_\ell) = \xi_\ell^*(g(\xi_\ell)) = \lambda(g) \cdot 1;$$

hence  $g\xi_\ell^* = \lambda(g)^{-1}\xi_\ell^*$  for every  $g \in G$ . It follows that

$$\kappa(g(\xi_i), g(\xi_j)) = \lambda(g) \cdot \kappa(\xi_i, \xi_j).$$

**PROPOSITION 5.2.** *Let  $A$  be a graded connected Gorenstein algebra, then the Brauer-character series of  $A$  and  $U$  satisfy the following identities:*

- (i)  $H_{U,g}^{Br}(t) = \hat{\lambda}(g) \cdot t^\beta H_{U,g^{-1}}^{Br}(1/t);$
- (ii)  $\frac{H_{A,g}^{Br}(t)}{H_{A,g^{-1}}^{Br}(1/t)} = (-1)^d t^{a(A)} \hat{\lambda}(g)$  with  $a(A) = \beta - \sum_i d_i = \deg(H_{A,1}^{Br}(t)).$

*In particular*

$$\hat{\lambda}(g) = (-1)^d \cdot \lim_{t \rightarrow 1} \frac{H_{A,g}^{Br}(t)}{H_{A,g^{-1}}^{Br}(1/t)}.$$

**REMARK 3.** It follows from (i) that the character  $\lambda$  only depends on  $A$  and not on the choice of  $\mathcal{F}$ . Therefore we denote it by  $\lambda_A$  and we will denote the corresponding Brauer character by  $\hat{\lambda}_A$ .

**Proof:** 1.: Let  $\mathfrak{A} := \{a_1, \dots, a_m\}$  and  $\mathfrak{B} := \{b_1, \dots, b_m\}$  be  $k$ -bases of  $U_i$  and  $U_{\beta-i}$ , respectively, then  $\kappa(g(a_i), g(b_j)) = \lambda(g) \cdot \kappa(a_i, b_j)$ . On the other hand, this is equal to  $M_{\mathfrak{A}}(g)^{tr} \circ Q \circ M_{\mathfrak{B}}(g)$ , where  $Q = \kappa(a_i, b_j) \in k^{m \times m}$ . For every  $0 \leq \nu \leq \beta$  with  $\nu \neq \beta - i$  we have

$$U_\nu \subseteq U_i^\perp := \{a \in U \mid \kappa(a, U_i) = 0\}.$$

Hence the map

$$U_i \times U_{\beta-i} \rightarrow k, \quad (a, b) \mapsto \kappa(a, b)$$

is a perfect pairing, in particular  $Q$  is a non-singular matrix. Therefore  $M_{\mathfrak{A}}(g)^{tr} = \lambda(g) \cdot Q \circ M_{\mathfrak{B}}(g)^{-1} \circ Q^{-1}$  and

$$\text{trace}(g|_{U_i}) = \lambda(g) \text{trace}(g|_{U_{\beta-i}}^{-1}),$$

from which 1. follows immediately.

2.: Using 1., the LHS is equal to

$$\frac{H_{U,g}^{Br}(t)}{H_{U,g^{-1}}^{Br}(1/t)} \cdot \frac{\prod_i (1 - t^{d_i})^{-1}}{\prod_i (1 - t^{-d_i})^{-1}} = \hat{\lambda}(g) \cdot t^{\beta - \sum_i d_i} (-1)^d.$$

◻

REMARK 4. Let  $g \in \text{GL}(V)$  semisimple,  $A := \text{Sym}(V^*) \cong k[x_1, \dots, x_n]$  with  $x_1, \dots, x_n$  a basis of  $V^*$ . We can assume that  $g(x_i) = \lambda_i x_i$  with eigenvalues  $\lambda_i \in U(k)$ , so with slight abuse of notation we obtain

$$H_{A,g}^{Br}(t) = \widehat{\text{trace}(g|_A)} = \prod_{i=1}^n (1 + \hat{\lambda}_i t + \hat{\lambda}_i^2 t^2 + \dots) = \prod_{i=1}^n \frac{1}{1 - \hat{\lambda}_i t} = \frac{1}{\widehat{\det(1 - tg)}}.$$

It follows that  $H_{A,g^{-1}}^{Br}(1/t) = \frac{1}{\widehat{\det(1 - g^{-1}1/t)}} =$

$$\frac{t^n}{\widehat{\det(t - g^{-1})}} = \frac{t^n \widehat{\det(g)}}{\widehat{\det(gt - 1)}} = (-1)^n t^n \widehat{\det(g)} \cdot \frac{1}{\widehat{\det(1 - gt)}} = (-1)^n t^n \widehat{\det(g)} \cdot H_{A,g}(t).$$

Hence  $\widehat{\lambda}_A(g) = \widehat{\det(g)}^{-1}$ .

PROPOSITION 5.3. Let  $A$  be a graded connected Gorenstein domain and also a factorial domain. Then  $\text{Hom}_{\mathcal{F}}(A, \mathcal{F}) \cong A\theta_A$  with  $\chi_A^{-1} = \lambda_A$  as defined in Remark 3. Moreover  $\text{Hom}_{\mathcal{F}}(R, \mathcal{F}) \cong A_\lambda$ , where

- (i)  $\lambda := \lambda_A$  if  $\text{char}(k)$  does not divide  $|G|$ ;
  - (ii)  $\lambda := \lambda_S \in \text{Hom}(G/W, U(k))$ , if  $S = A^W$  is Cohen-Macaulay (and therefore Gorenstein).
- In each of those cases  $R = A^G$  is quasi-Gorenstein if and only if  $\lambda = 1$ .

**Proof:** It follows from 1.2 that there exists some function  $\theta := \theta_A$  with  $\text{Hom}_{\mathcal{F}}(A, \mathcal{F}) \cong A \cdot \theta$ . From [4] Prop. 3.3.3 (a) we get

$$\overline{A} \cdot \overline{\theta} = \text{Hom}_{\mathcal{F}}(A, \mathcal{F}) \otimes \mathcal{F}/\mathcal{F}^+ \cong \text{Hom}_U(U, k) = U \cdot \xi_\ell^*,$$

hence  $\overline{\theta} = c \cdot \xi_\ell^*$  with some nonzero scalar  $c \in k$ . Setting  $\lambda := \lambda_A$ , it follows that  $\overline{g\theta} = g(\overline{\theta}) = \lambda(g)^{-1} \overline{\theta}$ , so  $g(\theta) - \lambda(g)^{-1} \cdot \theta \in \mathcal{F}^+ \text{Hom}_{\mathcal{F}}(A, \mathcal{F})$ . On the other hand  $G$  maps  $\theta$  onto another module generator and therefore  $g(\theta) = s_g \cdot \theta$  with a unit  $s_g \in k = A_0$ . It follows that  $g(\theta) - \lambda(g)^{-1} \cdot \theta \in k \cdot \theta \cap \mathcal{F}^+ A\theta = 0$  and we conclude  $g(\theta) = \lambda(g)^{-1} \cdot \theta$ . This shows  $\chi_A = \lambda_A^{-1}$ . Since  $S$  is a factorial domain, it is Gorenstein if Cohen-Macaulay, so the same argument as above gives  $\chi_S = \lambda_S^{-1}$ . The statement about  $\text{Hom}_{\mathcal{F}}(R, \mathcal{F})$  follows from 1.2.

For the rest of the proof we can assume that  $\text{char}(k)$  does not divide  $|G|$ . We consider the restriction map  $\text{res}: \text{Hom}_{\mathcal{F}}(A, \mathcal{F}) \rightarrow \text{Hom}_{\mathcal{F}}(R, \mathcal{F})$ ,  $\Psi \mapsto \Psi|_R$ . Since  $t: A \rightarrow R$ ,  $s \mapsto |G|^{-1} \sum_{g \in G} g(s)$  is an epimorphism of  $\mathcal{F}$ -modules and  ${}_{\mathcal{F}}R$  is free, we have  ${}_{\mathcal{F}}A = {}_{\mathcal{F}}R \oplus {}_{\mathcal{F}}X$  for some complement  ${}_{\mathcal{F}}X \subseteq A$ , hence  $\text{res}$  is surjective. Let  $\lambda \in \text{Hom}_{\mathcal{F}}(R, \mathcal{F})$ , then there is  $s \in A$  with  $\lambda = s \cdot \theta|_R = \theta(s \cdot \cdot)$ . For any  $r \in R$  we get

$$\lambda(r) = \frac{1}{|G|} \sum_{g \in G} \lambda(gr) = \frac{1}{|G|} \sum_{g \in G} \theta(sgr) = \frac{1}{|G|} \sum_{g \in G} \theta(g(g^{-1}(s)r)) =$$

$$\frac{1}{|G|} \sum_{g \in G} \lambda(g) \theta(g^{-1}(s)r) = \theta(t_\lambda(s)r),$$

where  $t_\lambda := \frac{1}{|G|} \sum_{g \in G} \lambda(g) g^{-1} : A \rightarrow A_\lambda$  is the projection operator in  $\text{Hom}_{\mathcal{F}}(A, A_\lambda)$ . Thus we have  $\text{Hom}_{\mathcal{F}}(R, \mathcal{F}) \subseteq \text{res}(A_\lambda \cdot \theta)$ . Again it follows from  ${}_{\mathcal{F}}A = {}_{\mathcal{F}}R \oplus {}_{\mathcal{F}}X$ , that  $\text{Hom}_{\mathcal{F}}(X, \mathcal{F})^G = 0$ , hence

$$\text{Hom}_{\mathcal{F}}(A, \mathcal{F})^G \cong \text{res}_R(\text{Hom}_{\mathcal{F}}(A, \mathcal{F})^G) = \text{Hom}_{\mathcal{F}}(A^G, \mathcal{F}).$$

Clearly  $A_\lambda \cdot \theta \in \text{Hom}_{\mathcal{F}}(A, \mathcal{F})^G$ , so

$$\text{Hom}_{\mathcal{F}}(R, \mathcal{F}) = \text{res}(A_\lambda \cdot \theta) \cong {}_{\mathcal{F}}A_\lambda.$$

If  $\lambda = 1$ , then  $\text{Hom}_{\mathcal{F}}(R, \mathcal{F}) = R \cdot \text{res}(\theta)$  is a cyclic  $R$ -module, so  $\omega_R \cong \text{Hom}_{\mathcal{F}}(R, \mathcal{F}) \cong {}_R R$  and  $R$  is Gorenstein.  $\circ$

**REMARK 5.** In the special case where  $A = \text{Sym}(V^*)$  with linear  $G$ -action the result above for the non-modular case also appears in [6] Cor. 3.2. The proof indicated there depends on the results of [7], [8]. In contrast to this our proof above is elementary and independent of Watanabe's results as well as of our Theorem 1.2.

One can apply the results above for example in the situation where  $A := \text{Sym}(V^*)$  for finite dimensional  $kG$ -module  $V$ , and  $S = A^W$  or  $S = A^{\tilde{W}}$ , with  $\tilde{G} := G/W$  or  $G/\tilde{W}$  acting on  $S$ . However, even if  $S = k[x_1, \dots, x_n]$  is a polynomial ring (with  $\deg(x_i) =: d_i \geq 1$ ), then action of  $\tilde{G}$  will in general be non-linear and the  $k$ -space  $\langle x_1, \dots, x_n \rangle_k$  will be not  $\tilde{G}$ -stable. Nevertheless we can use Remark 4 to determine  $\lambda_S = \chi_S^{-1}$ :

Let  $M$  be a finite dimensional  $kG$ -module with  $kG$ -submodule  $N \subseteq M$ . As a vectorspace we have  $M = N \oplus U$ , with  $U \cong M/N$  as a  $kG$ -module. Even though  $M$  and  $N \oplus U$  are in general not isomorphic as  $kG$ -modules, one has  $\chi_M = \chi_N + \chi_{M/N}$ . It follows that  $\text{Sym}(M) \cong \text{Sym}(N) \otimes_k \text{Sym}(U)$  as a  $k$ -algebra, but in general not as  $kG$ -module. Nevertheless we have  $H_{\text{Sym}(M),g}^{Br}(t) = H_{\text{Sym}(N),g}^{Br}(t) \cdot H_{\text{Sym}(M/N),g}^{Br}(t)$ . Even more generally, the following Lemma includes the case of a graded, but non-linear  $G$ -action on the algebra generators:

**LEMMA 5.4.** *Let  $G$  act on  $A$  by graded algebra automorphisms and  $B \leq A$  a  $G$ -stable graded subalgebra. Assume that  $A \cong B \otimes_k A/B_+A$  as a  $k$ -algebra (not necessarily as  $kG$ -module). Then*

$$H_{A,g}^{Br}(t) = H_{B,g}^{Br}(t) \cdot H_{A/B_+A,g}^{Br}(t).$$

**Proof:** Let  $A/B_+A =: C$  and identify the  $k$ -algebras  $B \otimes_k C \cong A$ , via  $b \otimes c = bc$ . Let  $x_1, \dots, x_\mu$  be a  $k$ -basis of  $B_m$  and  $y_1, \dots, y_\nu$  a  $k$ -basis of  $C_n$ . Then  $g(y_j) = \sum_t g_{C;tj} y_t + \mathfrak{B}\mathfrak{C}$  with  $\mathfrak{B}\mathfrak{C} \in \sum_{r=1}^n B_r C_{n-r}$  and the matrix  $(g_{C;tj})$  describing the representation of  $g$  on the  $kG$ -module  $C_j \cong (A/B_+A)_j$ . Hence  $g(x_i y_j) = g(x_i) g(y_j) =$

$$\sum_s (g_{B;si} x_s) \left( \sum_t (g_{C;tj} y_t + \mathfrak{B}\mathfrak{C}) \right) = g_{B;ii} \cdot g_{C;jj} \cdot x_i y_j + \sum_{(s,t) \neq (i,j)} g_{B;si} g_{C;tj} x_s y_t + \mathcal{X},$$

with  $\mathcal{X} \in \sum_{r=1}^n B_{m+r} C_{n-r} \subseteq B_+A$ . It follows that  $\chi_{A_{m+n},g} = \chi_{B_m,g} \cdot \chi_{C_n,g}$  and therefore  $H_{A,g}^{Br}(t) = H_{B,g}^{Br}(t) \cdot H_{C,g}^{Br}(t)$ .  $\circ$

PROPOSITION 5.5. Let  $A = k[x_{11}, \dots, x_{1j_1}, x_{21}, \dots, x_{2j_2}, \dots, x_{\ell 1}, \dots, x_{\ell j_\ell}]$  be a polynomial ring with generators of degrees  $1 \leq d_1 < d_2 < \dots < d_\ell$ . For  $i := 1, \dots, \ell$  let  $U_i$  denote the  $kG$ -module  $A_{d_i}/A^+A^+ \cap A_{d_i} \in kG\text{-mod}$  and  $\det_i : G \rightarrow k$ ,  $g \mapsto \det(g|_{U_i})$ . Then for every  $g \in G$ :

$$H_{A,g}^{Br}(t) = \prod_{i=1}^k H_{\text{Sym}(U_i),g}^{Br}(t) = \prod_{i=1}^k \frac{1}{\det(1 - t^{d_i}g)} \text{ and}$$

$$\widehat{\lambda}_A(g) = \prod_{i=1}^k \widehat{\det}_i(g)^{-1}.$$

**Proof:** The subalgebra  $B := k[x_{11}, \dots, x_{1j_1}] = \text{Sym}(U_1) \subseteq A$  is  $G$ -stable and we have  $A = B \otimes_k A/B^+A$  with polynomial ring  $A/B^+A \cong k[\overline{x}_{21}, \dots, \overline{x}_{2j_2}, \dots, \overline{x}_{\ell 1}, \dots, \overline{x}_{\ell j_\ell}]$ . Now the first equality follows from Lemma 5.4 and an obvious induction. The rest follows in a way similar to Remark 4.  $\circ$

If  $\text{char}(k) = p > 0$ , then by definition  $p$  does not divide  $[G : \tilde{W}]$ , hence if  $A^{\tilde{W}}$  is Cohen-Macaulay, so is  $A^G$ . With regard to the Gorenstein property we obtain the following:

COROLLARY 5.6. Let  $A := \text{Sym}(V^*)$  with finite dimensional  $kG$ -module  $V$  and assume that  $A^{\tilde{W}} \cong k[x_{11}, \dots, x_{1j_1}, x_{21}, \dots, x_{2j_2}, \dots, x_{\ell 1}, \dots, x_{\ell j_\ell}]$  is a polynomial ring with generators of degrees  $1 \leq d_1 < d_2 < \dots < d_\ell$ . Then  $A^G$  is Cohen-Macaulay and  $A^G$  is Gorenstein if and only if  $\prod_{i=1}^k \widehat{\det}_i(g)^{-1} = 1$  for all  $g \in G$ .

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